ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 323 (2006) 974–984

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Integral representations and convergence of the renewal density

Wolfgang Stadje

University of Osnabrück, Fachbereich Mathematik/Informatik, 49069 Osnabrück, Germany

Received 24 February 2005

Available online 7 December 2005

Submitted by U. Stadtmueller

Abstract

We derive integral representations for the renewal density u associated with a square integrable probability density p on $[0,\infty)$ having finite expected value μ. These representations express u in terms of the real and the imaginary part of the Fourier transform of p, considered as a function on the lower complex half plane. We use them to give simple global integrability conditions on p under which $\lim_{t\to\infty}(u(t)-p(t))=1/\mu$.

Keywords: Renewal density; Asymptotic behavior; Fourier transform; Integral representation

1. Introduction

Let p be a probability density on $[0,\infty)$ with finite expected value $\mu=\int_0^\infty xp(x)\,dx$. The renewal density associated with p is defined by

$$u(t)=\sum_{n=1}^{\infty}p_n(t),\qquad t\geqslant 0,$$

where p_n is the n fold convolution of p with itself. The asymptotic behavior of $u(t)$ is a classical topic which has been studied by many authors. The main question is of course whether

$$\lim_{t\to\infty}u(t)=1/\mu. \tag{1.1}$$

E-mail address: wolfgang@mathematik.uni-osnabrueck.de.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.11.003

Conditions for (1.1) to hold were given by Feller [2,3], Täcklind [10] and Smith [6–8]. The simplest are: (i) $\lim_{x\to\infty} p(x) = 0$ and $p \in L_{1+\delta}([0,\infty))$ for some $\delta > 0$ [6]; or (ii) p is directly Riemann integrable [3, p. 367]. Smith [8] provides sufficient and necessary conditions for (1.1).

If p is bounded, then

$$\lim_{t\to\infty} \bigl(u(t) - p(t)\bigr) = 1/\mu \tag{1.2}$$

[3, p. 367] which means that the renewal density asymptotically compensates the possible oscillations of p. More generally, if some convolution p_m is bounded, then

$$\lim_{t\to\infty} \left(u(t) - \sum_{n=1}^{m} p_n(t)\right) = 1/\mu \tag{1.3}$$

([3, Problem 7, p. 387] and [1, p. 92]). Estimates for the rate of convergence under various conditions have been given by Grübel [4,5] and Stadje [9].

In this paper we give simple global integrability conditions on p under which (1.2) is valid. We prove that (1.2) holds if $\int_0^\infty xp(x)\log(1+x)\,dx < \infty$ and $p \in L_2([0,\infty))$. The proof is based on certain new integral representations of $v(t) = u(t) - p(t)$ in terms of the real and imaginary parts of the Fourier transform $f(z) = \int_0^\infty e^{-izt} p(t)\,dt$, $\operatorname{Im} z \leqslant 0$. Write $f(z)$ in the form $f(x - ic) = l_1(x \mid c) - il_2(x \mid c)$, where the functions l_j are real-valued. Then we show that if $p \in L_2([0,\infty))$,

$$v(t) = \frac{2}{\pi} e^{ct} \int_0^\infty \frac{l_1^2 - l_2^2 - l_1^3 - l_1 l_2^2}{(1-l_1)^2 + l_2^2} \cos xt\,dx, \tag{1.4}$$

$$v(t) = \frac{2}{\pi} e^{ct} \int_0^\infty \frac{2l_1 l_2 - l_1^2 l_2 - l_2^3}{(1-l_1)^2 + l_2^2} \sin xt\,dx \tag{1.5}$$

for every $c > 0$, where $l_j = l_j(x \mid c)$. Interestingly, it is possible to set $c = 0$ in (1.5) but not in (1.4). Thus we obtain the identity

$$v(t) = \frac{2}{\pi} \int_0^\infty \frac{2f_1(x)f_2(x) - f_1(x)^2 f_2(x) - f_2(x)^3}{(1-f_1(x))^2 + f_2(x)^2} \sin xt\,dx, \tag{1.6}$$

where $f_j(x) = l_j(x \mid 0)$, $j = 1, 2$, i.e., $f(x) = f_1(x) - if_2(x)$. Our result on $\lim_{t\to\infty} v(t)$ follows from (1.6) after some analytic work.

Section 2 provides some Fourier analytic preliminaries. In Section 3 we derive the integral representations (1.4)–(1.6). Section 4 then gives the asymptotic result on $v(t)$.

2. Preliminaries

The Fourier transform

$$f(z) = \int_0^\infty e^{-izt} p(t)\,dt$$

of p is an analytic function in the lower complex half plane $\operatorname{Im} z < 0$ and continuous in its closure $\operatorname{Im} z \leqslant 0$. Since $|f(z)| < 1$ for $\operatorname{Im} z < 0$, the transform

$$g(z) = \int_0^\infty e^{-izt} u(t)\,dt = \sum_{n=1}^\infty f(z)^n$$

is also analytic in $\operatorname{Im} z < 0$, and we have

$$g(z) = f(z) + f(z)g(z), \quad \operatorname{Im} z < 0. \tag{2.1}$$

(2.1) corresponds to the standard renewal equation $u = p + u * p$ satisfied by u. If u is additionally continuous, the inversion formula yields

$$u(t) = \frac{1}{2\pi} \int_{-ic-\infty}^{-ic+\infty} g(z) e^{izt}\,dz, \quad c > 0. \tag{2.2}$$

The renewal density u is continuous if and only if p is continuous. So (2.2) can be directly applied only in this case. However, $v = u - p$ has the transform

$$h(z) = \int_0^\infty e^{-izt} v(t)\,dt = \frac{f(z)}{1 - f(z)} - f(z) = \frac{f(z)^2}{1 - f(z)}. \tag{2.3}$$

In order to use the inversion formula for h and v and to establish that v is continuous, we must ensure that $\int_{-ic-\infty}^{-ic+\infty} |h(z)|\,dz$ is finite. For this it is sufficient that $\int_{-ic-\infty}^{-ic+\infty} |f(z)|^2\,dz$ is finite; note that $f(x - ic) \neq 1$ and $\lim_{|x|\to\infty} f(x - ic) = 0$. If $p(t)e^{-ct}$ (the Fourier transform of $f(x - ic)$) is square integrable for some $c \geqslant 0$, it follows from the Parseval identity that

$$\int_{-ic-\infty}^{-ic+\infty} \left|f(z)\right|^2 dz = \int_{-\infty}^{\infty} \left|f(x - ic)\right|^2 dx = 2\pi \int_0^\infty p(t)^2 e^{-2ct}\,dt < \infty. \tag{2.4}$$

Now assume that p is square integrable. Then we can use (2.4), v is continuous and (2.2) for v and h instead of u and g, respectively, can be applied. We have proved

Lemma 1. *If $p \in L_2([0, \infty))$, then*

$$v(t) = \frac{1}{2\pi} \int_{-ic-\infty}^{-ic+\infty} e^{izt} \frac{f(z)^2}{1 - f(z)}\,dz \tag{2.5}$$

for any $c > 0$.

3. Real integral formulas for v

We introduce the real and the imaginary part of $f(x - ic)$, $x \in \mathbb{R}$, $c > 0$:

$$f(x - ic) = l_1(x \mid c) - il_2(x \mid c),$$

where

$$l_1(x \mid c) = \int_0^\infty e^{-ct} p(t) \cos xt \, dt, \tag{3.1}$$

$$l_2(x \mid c) = \int_0^\infty e^{-ct} p(t) \sin xt \, dt. \tag{3.2}$$

We can represent $v(t)$ in terms of $l_1(x \mid c)$ and $l_2(x \mid c)$ as follows.

Lemma 2. *Let $l_j = l_j(x \mid c)$, $j = 1, 2$. If $p \in L_2([0, \infty))$, then for every $c > 0$,*

$$v(t) = \frac{2}{\pi} e^{ct} \int_0^\infty \frac{l_1^2 - l_2^2 - l_1^3 - l_1 l_2^2}{(1 - l_1)^2 + l_2^2} \cos xt \, dx, \tag{3.3}$$

$$v(t) = \frac{2}{\pi} e^{ct} \int_0^\infty \frac{2l_1 l_2 - l_1^2 l_2 - l_2^3}{(1 - l_1)^2 + l_2^2} \sin xt \, dx. \tag{3.4}$$

Proof. Using (2.6) a short calculation yields

$$h(x - ic) = h_1(x \mid c) - ih_2(x \mid c),$$

where

$$h_1(x \mid c) = \frac{l_1^2 - l_2^2 - l_1^3 - l_1 l_2^2}{(1 - l_1)^2 + l_2^2},$$

$$h_2(x \mid c) = \frac{2l_1 l_2 - l_1^2 l_2 - l_2^3}{(1 - l_1)^2 + l_2^2}.$$

Changing variables in (2.8) we find that

$$\begin{aligned} v(t) &= \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} e^{ixt} \frac{f(x - ic)^2}{1 - f(x - ic)} \, dx \\ &= \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} \big(h_1(x \mid c) - ih_2(x \mid c)\big)(\cos xt + i \sin xt) \, dx \\ &= \frac{e^{ct}}{2\pi} \left[\int_{-\infty}^{\infty} (h_1 \cos xt + h_2 \sin xt) \, dx + i \int_{-\infty}^{\infty} (h_1 \sin xt - h_2 \cos xt) \, dx \right]. \end{aligned} \tag{3.5}$$

$h_1(x \mid c)$ and $h_2(x \mid c)$ are even and odd functions of x, respectively. Thus the first integrand on the right-hand side of (3.5) is even and the second one is odd. Hence,

$$\int_0^\infty (h_1 \cos xt + h_2 \sin xt) \, dx = \pi e^{-ct} v(t) \tag{3.6}$$

and, using that $v(-t) = 0$ for $t > 0$,

$$\int_0^\infty (h_1 \cos xt - h_2 \sin xt)\, dx = 0. \tag{3.7}$$

Now (3.3)–(3.4) follow easily from (3.6)–(3.7). □

We want to use (3.3) and (3.4) for $c = 0$, that is, let $c \searrow 0$ and interchange passing to the limit and integrating. It turns out that this is possible in (3.4) but not in (3.3).

Example. Let $p(t) = e^{-t}$. Then $v(t) = 1 - e^{-t}$, $f(x - ic) = (c + 1 - ix)/(x^2 + (c + 1)^2)$, and a little calculation yields

$$h_1(x \mid c) \cos xt = \frac{c(c+1) - x^2}{[c(c+1) - x^2]^2 + [(2c+1)x]^2} \cos xt$$

for the integrand in (3.3). Thus

$$h_1(x \mid 0) \cos xt = -\frac{\cos xt}{1 + x^2}.$$

Replacing the integrand in (3.3) by $h_1(x \mid 0) \cos xt$ yields

$$-\frac{2}{\pi} \int_0^\infty \frac{\cos xt}{1 + x^2}\, dx = -e^{-t} \neq v(t).$$

Now we show that (3.4) also holds for $c = 0$. Let

$$f_1(x) = l_1(x \mid 0) = \int_0^\infty p(t) \cos xt\, dt, \qquad f_2(x) = l_2(x \mid 0) = \int_0^\infty p(t) \sin xt\, dt.$$

Theorem 1. *If $p \in L_2([0, \infty))$, then for all $t > 0$,*

$$v(t) = \frac{2}{\pi} \int_0^\infty \frac{2 f_1(x) f_2(x) - f_1(x)^2 f_2(x) - f_2(x)^3}{(1 - f_1(x))^2 + f_2(x)^2} \sin xt\, dx. \tag{3.8}$$

Proof. We first consider the behavior of the parts of the integral in (3.4) at infinity and at zero and show:

(a) For every $\varepsilon > 0$ there are $K > 0$ and $C > 0$ such that

$$\sup_{0 \leqslant c \leqslant C} \int_K^\infty \left| h_2(x \mid c) \right| dx \leqslant \varepsilon.$$

(b) For every $t > 0$ and every $\varepsilon > 0$ there are $\alpha > 0$ and $C > 0$ such that

$$\sup_{0 \leqslant c \leqslant C} \int_0^\alpha \left| h_2(x \mid c) \sin xt \right| dx \leqslant \varepsilon.$$

To prove (a), take the derivative of l_j with respect to c. By (3.1)–(3.2),

$$\frac{\partial l_1(x \mid c)}{\partial c} = \left| \int_0^\infty t e^{-ct} p(t) \cos xt \, dt \right| \leqslant \mu,$$

$$\frac{\partial l_2(x \mid c)}{\partial c} = \left| \int_0^\infty t e^{-ct} p(t) \sin xt \, dt \right| \leqslant \mu.$$

Thus,

$$\left| l_j(x \mid c) - f_j(x) \right| \leqslant \mu c, \quad j = 1, 2,$$

so that $l_j(x \mid c)$ converges to $f_j(x)$ uniformly with respect to x, as $c \searrow 0$. As $\lim_{|x|\to\infty} f_j(x) = 0$ by the Riemann–Lebesgue lemma, it easily follows that there are constants $\eta, C_0, K_0 > 0$ such that

$$\left(1 - l_1(x \mid c)\right)^2 + l_2(x \mid c)^2 \geqslant \eta \quad \text{for } (x, c) \in [K_0, \infty) \times [0, C_0], \tag{3.9}$$

i.e., the denominator in (3.4) remains bounded away from zero in this (x, c)-region. Moreover,

$$\lim_{c \searrow 0} \int_I \left| f(x - ic) \right|^2 dx = \int_I \left| f(x) \right|^2 dx \tag{3.10}$$

for all compact intervals $I \subset \mathbb{R}$. By (2.7) and the monotone convergence theorem,

$$\lim_{c \searrow 0} \int_{-\infty}^{\infty} \left| f(x - ic) \right|^2 dx = \lim_{c \searrow 0} 2\pi \int_0^\infty p(t)^2 e^{-2ct} \, dt = \int_0^\infty p(t)^2 \, dt = \int_{-\infty}^{\infty} \left| f(x) \right|^2 dx. \tag{3.11}$$

Since $|f(x - ic)|^2$ is an even function of x, we can conclude from (3.10) and (3.11) that

$$\lim_{c \searrow 0} \int_T^\infty \left| f(x - ic) \right|^2 dx = \int_T^\infty \left| f(x) \right|^2 dx \quad \text{for all } T > 0. \tag{3.12}$$

The right-hand side of (3.12) tends to zero as $T \to \infty$. Hence, for every $\varepsilon > 0$ there are $c_1 > 0$ and $K_1 > 0$ such that

$$\int_{K_1}^\infty \left| f(x - ic) \right|^2 dx < \varepsilon \quad \text{for all } c \in [0, c_1]. \tag{3.13}$$

The same is true if $|f(x - ic)|^2$ in (3.13) is replaced by one of the integrands

$$\left| l_1(x \mid c) l_2(x \mid c) \right|, \quad \left| l_1(x \mid c) \right|^2 \left| l_2(x \mid c) \right|, \quad \left| l_2(x \mid c) \right|^3,$$

which are not greater than $|f(x - ic)|^2$; note that $|f(x - ic)|$ is bounded by 1. Combining this with the lower estimate (3.9) for the denominator of $h_2(x \mid c)$, we obtain a proof of assertion (a).

Now we turn to (b). Write the integrand $|h_2(x \mid c) \sin xt|$ as

$$\left| h_2(x \mid c) \sin xt \right| = \left| 2l_1 - l_1^2 - l_2^2 \right| \left| \frac{l_2^2}{(1 - l_1)^2 + l_2^2} \right| \left| \frac{x}{l_2} \right| \left| \frac{\sin xt}{x} \right|. \tag{3.14}$$

Fix t and consider the four factors on the right-hand side of (3.14) as functions of $(x, c) \in (0, \infty) \times [0, \infty)$. The first, the second and the fourth factor are obviously bounded. We show that the third factor is bounded in some neighborhood of $(0, 0)$. For a given $\delta > 0$ choose $u_\delta > 0$ and $M_\delta > 0$ such that $1 - \delta \leqslant u^{-1} \sin u \leqslant 1$ for $u \in (0, u_\delta]$ and $\int_{M_\delta}^\infty tp(t)\,dt < \delta/2$. Then clearly, for $x \in (0, u_\delta/M_\delta)$,

$$
\begin{aligned}
0 &\leqslant \int_0^\infty te^{-ct} p(t)\,dt - \frac{l_2(x \mid c)}{x} = \int_0^\infty te^{-ct} p(t)\left(1 - \frac{\sin xt}{xt}\right) dt \\
&\leqslant \int_0^{M_\delta} tp(t)\delta\,dt + 2\int_{M_\delta}^\infty tp(t)\,dt \\
&\leqslant (\mu + 1)\delta.
\end{aligned}
$$

Since $\int_0^\infty te^{-ct} p(t)\,dt \to \mu$ as $c \searrow 0$, it follows that there are constants $\alpha > 0$, $C > 0$ such that $|l_2(x \mid c)/x|$ is bounded away from zero for $(x, c) \in (0, \alpha] \times [0, C]$. Thus also the third factor in (3.14) is bounded in this (x, c)-region. This yields assertion (b).

From (3.4), (a) and (b) we conclude that for every $t > 0$ and $\varepsilon > 0$ there are $c_\varepsilon > 0$, $\delta_\varepsilon > 0$ and $K_\varepsilon > \delta_\varepsilon$ such that

$$
\left| v(t) - \frac{2}{\pi} e^{ct} \int_{\delta_\varepsilon}^{K_\varepsilon} \frac{2l_1(x \mid c)l_2(x \mid c) - l_1(x \mid c)^2 l_2(x \mid c) - l_2(x \mid c)^3}{(1 - l_1(x \mid c))^2 + l_2(x \mid c)^2} \sin xt\,dx \right| \leqslant \varepsilon \tag{3.15}
$$

for all $c \in (0, c_\varepsilon]$. Note that we do not yet know if (3.15) also holds for $c = 0$. For $x \in [\delta_\varepsilon, K_\varepsilon]$ the denominator of the integrand is positive, and as $c \searrow 0$ the two functions $l_j(x \mid c)$ converge to $l_j(x \mid 0) = f_j(x)$ uniformly in x. Thus, (3.15) is also valid for $c = 0$. Since the statements (a) and (b) include the value $c = 0$, it follows that

$$
\left| v(t) - \frac{2}{\pi} \int_0^\infty \frac{2l_1(x \mid 0)l_2(x \mid 0) - l_1(x \mid 0)^2 l_2(x \mid 0) - l_2(x \mid 0)^3}{(1 - l_1(x \mid 0))^2 + l_2(x \mid 0)^2} \sin xt\,dx \right| \leqslant \varepsilon \tag{3.16}
$$

holds for every $\varepsilon > 0$. This proves the theorem. □

4. Asymptotic behavior

Our result on the limiting behavior of $v(t)$ is based on the following asymptotic simplification of Theorem 1.

Lemma 3. *If $p \in L_2([0, \infty))$,*

$$
\lim_{t\to\infty} \left| v(t) - \frac{2}{\pi} \int_0^a \frac{f_2(x)}{(1 - f_1(x))^2 + f_2(x)^2} \sin xt\,dx \right| = 0 \quad \textit{for every } a > 0. \tag{4.1}
$$

Proof. $h_2(x \mid 0)$ is a continuous function on every compact interval $[a, b] \subset (0, \infty)$. Thus,

$$
\lim_{t\to\infty} \int_a^b h_2(x \mid 0) \sin xt\,dx = 0
$$

by the Riemann–Lebesgue lemma. Together with assertion (a) in Section 3 this yields

$$\lim_{t\to\infty}\int_a^\infty h_2(x\mid 0)\sin xt\,dx = 0 \quad \text{for every } a>0. \tag{4.2}$$

Next note that

$$\frac{2f_1(x)f_2(x) - f_1(x)^2 f_2(x) - f_2(x)^3}{(1-f_1(x))^2 + f_2(x)^2} = \frac{f_2(x)}{(1-f_1(x))^2+f_2(x)^2} - f_2(x)$$

so that

$$\int_0^a h_2(x\mid 0)\sin xt\,dx = \int_0^a \frac{f_2(x)}{(1-f_1(x))^2+f_2(x)^2}\sin xt\,dx - \int_0^a f_2(x)\sin xt\,dx. \tag{4.3}$$

As $f_2(x)$ is continuous on $[0,\infty)$, it is clear that $\lim_{t\to\infty}\int_0^a f_2(x)\sin xt\,dx = 0$. Relation (4.1) now follows from Theorem 1, (4.2) and (4.3). $\square$

We can rewrite the integral in (4.1) as follows. Let $F(x)$ be the distribution function corresponding to p and

$$F_1(x) = x^{-1}\bigl(1-f_1(x)\bigr) = \int_0^\infty \bigl(1-F(t)\bigr)\sin xt\,dt,$$

$$F_2(x) = x^{-1}f_2(x) = \int_0^\infty \bigl(1-F(t)\bigr)\cos xt\,dt.$$

Then

$$\int_0^a \frac{f_2(x)}{(1-f_1(x))^2+f_2(x)^2}\sin xt\,dx = \int_0^a \frac{F_2(x)}{F_1(x)^2+F_2(x)^2}\frac{\sin xt}{x}\,dx. \tag{4.4}$$

Note that F_1 and F_2 are continuous, $\mu \geqslant F_2(x)$ for all $x\geqslant 0$, $\lim_{x\searrow 0}F_1(x)=0$, $\lim_{x\searrow 0}F_2(x)=\mu$ and $\lim_{x\searrow 0}F_2(x)/(F_1(x)^2+F_2(x)^2) = 1/\mu$. We are now ready to prove

Theorem 2. *If $p\in L_2([0,\infty))$ and $\int_0^\infty xp(x)\log(1+x)\,dx<\infty$, then*

$$\lim_{t\to\infty} v(t) = \mu^{-1}.$$

Proof. It is easy to check that

$$\frac{F_2}{F_1^2+F_2^2} = \frac{1}{\mu} + \frac{\mu - F_2}{\mu F_2} - \frac{F_1^2}{F_2(F_1^2+F_2^2)}$$

so that the right-hand side of (4.4) can be decomposed into three integrals:

$$\begin{aligned}
&\int_0^a \frac{F_2(x)}{F_1(x)^2+F_2(x)^2}\frac{\sin xt}{x}\,dx \\
&\quad = \frac{1}{\mu}\int_0^a \frac{\sin xt}{x}\,dx + \int_0^a \frac{\mu-F_2(x)}{x}\frac{\sin xt}{\mu F_2(x)}\,dx - \int_0^a \frac{F_1(x)^2}{x}\frac{\sin xt}{F_2(x)(F_1(x)^2+F_2(x)^2)}\,dx \\
&\quad = I_1(t,a)+I_2(t,a)-I_3(t,a).
\end{aligned} \tag{4.5}$$

We will now show that

$$I = \int_0^1 \frac{\mu-F_2(x)}{x}\,dx < \infty \tag{4.6}$$

and

$$II = \int_0^1 \frac{F_2(x)^2}{x}\,dx < \infty. \tag{4.7}$$

(4.6)–(4.7) imply that

$$I(a) = \int_0^a \frac{\mu-F_2(x)}{x}\,dx \to 0, \quad \text{as } a \searrow 0, \tag{4.8}$$

and

$$II(a) = \int_0^a \frac{F_1(x)^2}{x}\,dx \to 0, \quad \text{as } a \searrow 0. \tag{4.9}$$

Now suppose that (4.6) and (4.7), and thus also (4.8) and (4.9), hold. Let $\varepsilon > 0$ be arbitrary. Choose $a_0 > 0$ such that

$$F_2(x) \geqslant \mu/2 \quad \text{for all } x \in [0,a_0], \qquad I(a_0) < \varepsilon, \qquad II(a_0) < \varepsilon.$$

Then

$$\begin{aligned}
\left|I_2(t,a_0)\right| &\leqslant I(a_0)\frac{1}{\mu^2/2} < \varepsilon\frac{2}{\mu^2}, \\
\left|I_3(t,a_0)\right| &\leqslant II(a_0)\frac{1}{(\mu/2)^3} < \varepsilon\frac{8}{\mu^3}.
\end{aligned}$$

Moreover,

$$\lim_{t\to\infty} I_1(t,a_0) = \frac{\pi}{2\mu}.$$

Using these estimates and (4.1) and (4.4) for $a = a_0$, we obtain

$$\limsup_{t\to\infty}\left|v(t)-\mu^{-1}\right| \leqslant \limsup_{t\to\infty}\left|v(t)-\frac{2}{\pi}\int_0^{a_0}\frac{F_2(x)}{F_1(x)^2+F_2(x)^2}\frac{\sin xt}{x}\,dx\right|$$
$$+\limsup_{t\to\infty}\left|\frac{2}{\pi}\int_0^{a_0}\frac{F_2(x)}{F_1(x)^2+F_2(x)^2}\frac{\sin xt}{x}\,dx-\mu^{-1}\right|$$
$$<\varepsilon\frac{2}{\pi}\left(\frac{2}{\mu^2}+\frac{8}{\mu^3}\right). \tag{4.10}$$

Since (4.10) holds for every $\varepsilon > 0$, the result follows.

Finally, let us prove (4.6) and (4.7). We can estimate I as follows:

$$\begin{aligned} I &= \int_0^1\int_0^\infty \big(1-F(t)\big)\frac{1-\cos tx}{x}\,dt\,dx \\ &= \int_0^\infty \big(1-F(t)\big)\int_0^1 \frac{1-\cos tx}{x}\,dx\,dt \\ &= \int_0^\infty \big(1-F(t)\big)\int_0^t \frac{1-\cos u}{u}\,du\,dt \\ &< \int_0^\infty \big(1-F(t)\big)\int_0^t \frac{3}{1+u}\,du\,dt \\ &= 3\int_0^\infty \big(1-F(t)\big)\log(1+t)\,dt \end{aligned} \tag{4.11}$$

and the right-hand side is finite by assumption. In (4.11) we have used Fubini's theorem and the inequality $u^{-1}(1-\cos u) < 3/(1+u)$ (which is valid for all $u > 0$).

Regarding II, we write it as

$$\begin{aligned} II &= \int_0^1 \frac{1}{x}\left[\int_0^\infty\int_0^\infty \big(1-F(t)\big)\big(1-F(u)\big)\sin tx\sin ux\,dt\,du\right]dx \\ &= \frac{1}{2}\int_0^1\left[\int_0^\infty\int_0^\infty\left(\big(1-F(t)\big)\big(1-F(u)\big)\frac{1-\cos(t+u)x}{x}\right.\right. \\ &\quad \left.\left.-\big(1-F(t)\big)\big(1-F(u)\big)\frac{1-\cos(t-u)x}{x}\right)dt\,du\right]dx. \end{aligned} \tag{4.12}$$

Proceeding similarly as in (4.11) it is seen that

$$\int_0^1\int_0^\infty\int_0^\infty \big(1-F(t)\big)\big(1-F(u)\big)\frac{1-\cos(t-u)x}{x}\,dt\,du\,dx$$

$$
\begin{aligned}
&< 3\int_0^\infty\int_0^\infty \bigl(1-F(t)\bigr)\bigl(1-F(u)\bigr)\log\bigl(1+|t-u|\bigr)\,dt\,du \\
&= 6\int_0^\infty\int_0^\infty \bigl(1-F(s+v)\bigr)\bigl(1-F(v)\bigr)\log(1+s)\,ds\,dv \\
&\leqslant 6\int_0^\infty\int_0^\infty \bigl(1-F(s)\bigr)\bigl(1-F(v)\bigr)\log(1+s)\,ds\,dv \\
&= 6\mu\int_0^\infty \bigl(1-F(s)\bigr)\log(1+s)\,ds \\
&< \infty. \qquad (4.13)
\end{aligned}
$$

The part of the right-hand side of (4.12) containing $\cos(t+u)x$ is handled in the same way. It remains to show that

$$\int_0^\infty\int_0^\infty \bigl(1-F(t)\bigr)\bigl(1-F(u)\bigr)\log\bigl(1+(t+u)\bigr)\,dt\,du < \infty. \qquad (4.14)$$

The integrand in (4.14) is bounded by

$$A(t,u) = \bigl(1-F(t)\bigr)\bigl(1-F(u)\bigr)\bigl[\log(1+t)+\log(1+u)\bigr]$$

and

$$\int_0^\infty\int_0^\infty A(t,u)\,dt\,du \leqslant 2\mu\int_0^\infty \bigl(1-F(t)\bigr)\log(1+t)\,dt < \infty.$$

The proof is complete. □

Acknowledgment

I thank the referee for his or her careful reading of the manuscript.

References

[1] G. Alsmeyer, Erneuerungstheorie, Teubner, Stuttgart, 1991.
[2] W. Feller, On the integral equation of renewal theory, Ann. Math. Statist. 12 (1941) 243–267.
[3] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2, second ed., Wiley, New York, 1971.
[4] R. Grübel, Über das aymptotische Verhalten von Erneuerungsdichten, Math. Nachr. 108 (1982) 39–48.
[5] R. Grübel, Stochastic models as functionals: Some remarks on the renewal case, J. Appl. Probab. 26 (1989) 296–303.
[6] W.L. Smith, Asymptotic renewal theorems, Proc. Roy. Soc. Edinburgh Sect. A 64 (1954) 9–48.
[7] W.L. Smith, Extensions of a renewal theorem, Proc. Cambridge Philos. Soc. 51 (1954) 629–638.
[8] W.L. Smith, On necessary and sufficient conditions for the convergence of the renewal density, Trans. Amer. Math. Soc. 104 (1962) 79–100.
[9] W. Stadje, The asymptotic behavior of the renewal density, Methods Oper. Res. 58 (1989) 307–317.
[10] S. Täcklind, Fourieranalytische Behandlung vom Erneuerungsproblem, Scand. Actuar. J. 28 (1945) 68–105.